Note on waves through gases at pressures small compared with the magnetic pressure, with applications to upper-atmosphere aerodynamics

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Most treatments of magnetohydrodynamic waves have confined physical interpretation to cases when the Alfvén velocity a_1 is small compared with the sound velocity a_0. Here we consider the 'low-beta situation', in which a_1 is much larger than a_0. Then, except for two modes with wave velocity a_1, the only possible waves are longitudinal ones, propagated unidirectionally along lines of magnetic force with velocity a_0. These can be interpreted as sound waves, confined to effectively rigid magnetic tubes of force. Hall-current effects do not alter these conclusions (in contrast to the high-beta situation), and finite conductivity introduces only small dissipation.

An application is made to the flow pattern around a body moving through the F_2 layer of the ionosphere, where, although neutral particles have a very large mean free path, charged particles interact electrostatically and, it is argued, may be regarded as forming a continuous fluid whose movement is independent of that of the neutral particles. A body moving at satellite speed or below would then excite the above-mentioned unidirectional sound waves, but no waves at much faster Alfvén velocity. These considerations suggest that its movement would be accompanied by a V-shaped pattern of electron density (figure 2), which might be in part responsible for some anomalous radar echoes that have been reported.

1. Introduction

Radar observation of satellites and other bodies moving through the F_2 layer of the ionosphere has sometimes, although not always, detected an extraordinarily large response (see, for example, Kraus, Higgy, Schees & Crone 1960). To investigate possible explanations of this, it is necessary to consider how disturbances, due either to the motion of the body or to such gas as it may be giving out (even though its propulsive motors have shut down), are propagated within the layer.

Now, such customary aerodynamic concepts as the shock-wave pattern of a supersonic projectile, or ideas from free-molecule theory, need to be treated with caution in the upper ionosphere, because this is a region of conducting fluid in a magnetic field. The field, H, is typically 0·5 G in middle magnetic latitudes, say, where the angle of dip is around 60°. In this case the magnetic pressure $H^2/8\pi$ is 0·01 dyne/cm^2 (the electromagnetic stress system consisting, of course, of this isotropic magnetic pressure, together with a tension of double the amount along the magnetic lines of force). However, the gas pressure is 0·01 dyne/cm^2 already at a height of 130 km, and is much lower (of order 10^{-4} dynes/cm^2) in the F_2 layer (altitude around 300 km).

We must ask therefore: how is the propagation of disturbances through a gas affected by the gas pressure p being small compared with the magnetic pressure $H^2/8\pi$? Note that plasma physicists call this a 'low-beta situation', β being the ratio $8\pi p/H^2$. Note also that the answer would be expected to involve not only the speed of sound a_0, which equals $\sqrt{(\gamma p/\rho)}$ with γ between 1 and 2, but also the Alfvén magnetohydrodynamic wave velocity, $a_1 = H/\sqrt{(4\pi\rho)}$.

This represents, for an incompressible conducting fluid in a magnetic field, the speed of propagation of transverse waves along magnetic tubes of force, which carry an effective tension $H^2/4\pi$ per unit area. One can interpret the propagation velocity a_1 as the square root of the ratio of tension to mass per unit length, as for waves on a string. The assumption of incompressibility holds good when $a_1 \ll a_0$.

However, when $\beta = 8\pi p/H^2$ is very small, a_1 is actually much greater than a_0, by a factor of order $\beta^{-\frac{1}{2}}$. Under these circumstances the physical interpretation of Alfvén waves needs to be reconsidered, and this is done in § 2.

Another concept that needs to be reconsidered in the upper ionosphere (§ 3) is that of 'mean free path'. Although the mean free path, in the sense of the distance travelled by a neutral particle before it comes close enough to another particle (of any kind) to exchange momentum with it, is very large, of the order of kilometres, nevertheless, certain other lengths which have most of the essential properties of a mean free path are considerably smaller.

First, a charged particle will exchange momentum with another charged particle even though their distance apart is a million molecular diameters. Actually, the effective mean free path for momentum exchange by charged particles is about $10^{-1}(kT)^2/n_e e^4$, where n_e is the electron density and e the electronic charge in e.s.u. This quantity is about 400 m in the F_2 layer. Furthermore, the momentum exchange can be shown to take place almost continuously, by means of a large number of mostly very small exchanges spread out over this distance. The ions and electrons may therefore behave almost like a continuous fluid in relation to the motion of a large body, while the neutral particles behave like a free-molecule gas, and the two gases do not interact at all.

Secondly, the charged particles do not pursue a straight path between encounters, but rather spiral about the magnetic lines of force. The root-mean-square radius of the spiral for electrons of mass m_e and charge $-e$ is $(kTm_e)^{\frac{1}{2}}\,c/eH$ which is of the order of a centimetre. For ions of mass $16 \times 1848 m_e$ and charge $+e$ the said r.m.s. radius is of the order of a metre. Thus for charged particles, the movement at right angles to magnetic lines of force is greatly restricted. Motion along magnetic lines of force, on the other hand, will occur on the average to a distance equal to the 'mean free path for charged particles' before momentum in that direction is lost by encounters.

With this background, the character of the waves set up by a body moving through the F_2 layer is discussed in § 4. The author takes this opportunity to thank Dr A. G. Touch, Mr B. Burgess, Dr R. L. F. Boyd, Dr R. N. Cox and Mr D. G. King-Hele for discussions of the problem.

2. Magnetohydrodynamic waves for very small β

The author has rather fully discussed the mathematics of magnetohydrodynamic waves elsewhere (Lighthill 1960). Results from that paper will be quoted freely.

We note first the results for a perfectly conducting, compressible fluid in a uniform magnetic field. Then, if we take the x-axis along the magnetic lines of force, the motion is conveniently described not in terms of the velocity components u, v, w, but in terms of u,

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \quad \text{and} \quad \delta = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{1}$$

(Here we make a slight change from the cited paper, where $\Gamma = \partial u/\partial x$, ξ and $\Delta = \Gamma + \delta$ were used.)

In fact, ξ, which is the vorticity component along magnetic lines of force, satisfies the simple wave equation

$$\frac{\partial^2 \xi}{\partial t^2} = a_1^2 \frac{\partial^2 \xi}{\partial x^2}, \tag{2}$$

representing straightforward one-dimensional propagation along magnetic lines of force at the Alfvén velocity a_1.

The propagation of u and δ is more complicated, but, as it turns out, more important for our purpose. The following plane-wave solutions are possible for small disturbances:

$$\left.\begin{aligned} u &= u_0\, e^{i(\omega t - \alpha x - \beta y - \gamma z)}, \\ \delta &= i\left(\alpha - \frac{\omega^2}{a_0^2 \alpha}\right) u_0\, e^{i(\omega t - \alpha x - \beta y - \gamma z)}, \end{aligned}\right\} \tag{3}$$

where

$$\beta^2 + \gamma^2 = \frac{(\omega^2 - a_0^2\alpha^2)(\omega^2 - a_1^2\alpha^2)}{(a_0^2 + a_1^2)\,\omega^2 - a_0^2 a_1^2 \alpha^2}. \tag{4}$$

In the 'high-beta situation', $a_1 \ll a_0$, the wave-number surface (of revolution) defined by equation (4) is in three sheets as illustrated in figure 1. The nearly spherical sheet represents sound waves, propagated with velocity nearly a_0 in all directions. The nearly plane sheets represent Alfvén waves, propagated almost one-dimensionally along lines of magnetic force with velocity a_1. This is true even of waves with large β and γ, since the energy propagation velocity or group velocity is always in the direction of the normal to the wave-number surface. Equations (2) and (3) show that in sound waves, propagating with velocity a_0, the vorticity ξ must vanish; while in Alfvén waves ξ and δ, representing the transverse fluid motions, are non-zero, and the longitudinal motion u is coupled to them by the fact that $\partial u/\partial x + \delta$ is practically zero (incompressibility).

Turning now to the low-beta situation, with $a_0 \ll a_1$, we observe first that the equation (4) of the wave-number surface is completely unaltered if a_0 and a_1 be interchanged. Thus, figure 1 still applies, but the radius of the nearly spherical sheet is ω/a_1, and the distance of the nearly plane sheets from the origin is ω/a_0. The first fact means that in low-beta situations waves can be propagated in *all* directions with nearly equal velocity a_1. These waves carry mainly variations in the two-dimensional divergence δ; note that the vorticity ξ is also propagated with velocity a_1, but solely one-dimensionally, along magnetic lines of force. The second fact shows that waves are also propagated one-dimensionally, along magnetic lines of force, with the *sound* velocity, a_0.

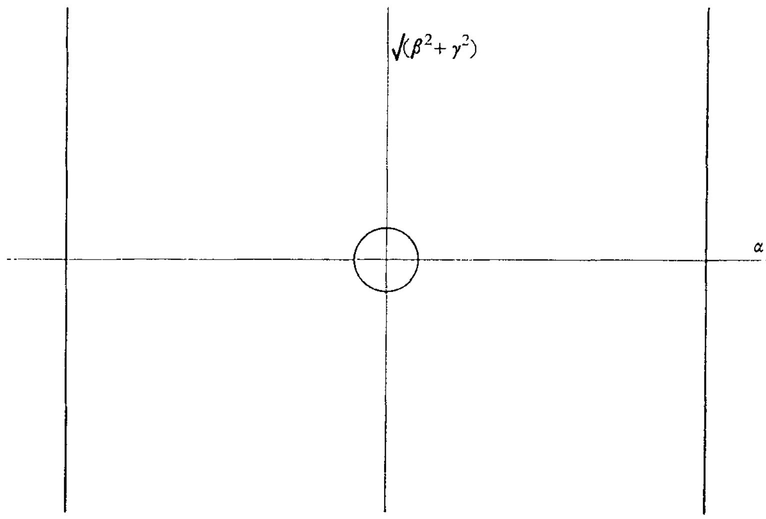

FIGURE 1. Shape of the wave-number surface given by equation (4) in the case when $a_1/a_0 = 0{\cdot}1$. The central 'sphere' is really slightly ovoid, with major axis ω/a_0 and minor axis shorter by 0·5 %. The 'plane' sheets are at distance ω/a_1 from the origin, and depart from exact planeness by 1 % of this distance. When $a_1/a_0 = 10$ instead of 0·1 the wave-number surface still has this shape, but the major axis of the central 'sphere' is then ω/a_1 and the 'plane' sheets are at distance ω/a_0 from the origin.

Now, in problems of propagation of the effect of missiles, the wave velocity a_1 turns out to be too great compared with the missile velocity for the waves propagated with velocity a_1 to be excited. Only the waves propagated with velocity a_0 are excited, and so it is of the greatest importance to determine their nature.

We note, first, that the mathematical theory for a perfectly conducting, compressible fluid shows them to be longitudinal waves (carrying non-zero values of u, but with $\xi = 0$, and also, owing to the factor $\alpha - \omega^2/a_0^2\alpha$ in (3), δ practically zero), which propagate one-dimensionally along magnetic lines of force. The physical explanation of this result is that in low-beta situations, when magnetic forces greatly exceed pressure forces, magnetic lines of force are stretched to so high a tension as to be practically rigid to sound waves, which therefore can only propagate longitudinally along them, with magnetic tubes of force behaving like speaking tubes.

This physical explanation helps us to see how far the same result will be true for a less special fluid. The effect of finite conductivity is unlikely to be great, because the longitudinal motions induce no current. They do not, in fact, deform the magnetic lines of force at all. However, the mechanism for keeping the motion longitudinal involves some small currents, since any pressure gradient at right angles to magnetic lines of force (say, in the y-direction) will immediately relieve itself by generating a current in the z-direction, such that the '$\mathbf{j} \times \mathbf{H}$' force on the current balances the pressure gradient—the time required for generation of this current being small (of order β) in low-beta situations, compared with that for passage of a wave at velocity a_0. Joule heating by these currents would have some dissipative effect, but they are weak when H is large relative to grad p.

Also, Hall effect should be negligible, which is in very striking contrast to the high-beta situation (Lighthill 1960) and is of importance in the ionosphere, where the usual condition $B > 10^{-6} n_e T^{-\frac{3}{2}}$ for Hall effect to exceed Ohm effect in importance is satisfied with a factor of 10^4 to spare. The reason is that Hall effect is equivalent to an addition to the electric field of a term proportional to $\mathbf{j} \times \mathbf{H}/n_e$ and therefore to $(\text{grad}\, p)/n_e$, so that its contribution to $\partial \mathbf{H}/\partial t$ via curl $\mathbf{E}$ is zero, at least if the electron density n_e varies in direct functional relationship to the pressure.

3. Applicability of the theory to the upper ionosphere

We consider next whether the upper ionosphere is in fact dense enough to be treated as a fluid at all. The mean free path for neutral atoms and molecules is of the order of $10^{-8}/\rho$ in c.g.s. units, and at densities of $10^{-13}\,\text{g/cm}^3$ (corresponding to 250 km altitude) is already 1 km. However, the mean free path is smaller for charged particles, for in propagation problems of the kind we are considering it is defined in relation to momentum, as the average distance travelled by a particle before its momentum component in the direction of its existing momentum falls to zero. This does not, in the case of charged particles, require any collision to take place.

In fact, the kinetic theory of ionized gases shows that the mean free path of both ions and electrons in this sense (Ferraro 1933; Spitzer 1956) is

$$\frac{(kT)^2}{A n_e e^4}, \tag{5}$$

where A is a factor which depends logarithmically on a number of variables, but in practice is about 10. With $n_e = 10^6$ electrons/cm^3, corresponding to 300 km altitude, where the mean free path of neutral particles (number density about 3×10^9) is 3 km, and $T \simeq 1100\,^\circ\text{K}$, the mean free path of charged particles is about 400 m.

Furthermore, the factor A is logarithmic (Spitzer 1956) because the rate of loss of directed momentum is obtained by considering encounters at different separations, and integrating up the associated loss rate, which varies as the inverse power of the separation, the limit of integration with respect to the separation being the 'cut-off' value given by the Debye length

$$(kT/4\pi n_e e^2)^{\frac{1}{2}} \simeq 0{\cdot}07\ \text{cm}.$$

Accordingly, a large contribution to the total rate of loss of directed momentum comes from the separation ranges 10^{-4} to 10^{-3} cm, 10^{-3} to 10^{-2} cm and 10^{-2} to 10^{-1} cm, encounters within which occur on the average every time the particle travels 30, 0·3 and 0·003 cm, respectively. Thus, large parts of the momentum transfer occur in a manner which can be regarded as continuous for phenomena with a length scale of the order of metres.

It is also to be noted that particles do not traverse their mean free path in anything like a straight line. Their movement at right angles to magnetic lines of force is constrained to tight spirals of root-mean-square radius $(kTm)^{\frac{1}{2}} c/eH$, where m is the mass. For $T = 1100\,^{\circ}\mathrm{K}$ this is 1·5 cm for electrons and 2·6 m for O^+ ions.

We see, then, that for disturbances whose length scale exceeds 1 m, but is small compared with 3 km, the charged particles may behave as if they made up by themselves a continuous fluid, whereas the uncharged particles exhibit a completely independent, free-molecule behaviour.

The density of the ion-electron gas, corresponding to $n_e = 10^6\,\mathrm{cm}^{-3}$, is $\rho = 3 \times 10^{-17}\,\mathrm{g/cm}^3$, and the appropriate Alfvén velocity $a_1 = H/\surd(4\pi\rho)$ is therefore about 300 km/sec, compared with a sound speed of about 1 km/sec. Accordingly, bodies at the satellite speed of 8 km/sec, or less, can excite sound waves but not Alfvén waves.* At the same time, owing to the substantial mean free path $\lambda \simeq 4 \times 10^4$ cm, the sound waves that are produced will be diffused as if by a large kinematic viscosity $\nu \doteqdot \frac{2}{3} a_0 \lambda$.

Accordingly, the problem reduces to that of the propagation of large-amplitude one-dimensional acoustic disturbances through a highly viscous gas. This problem is one which happens to have been treated at length by the author (1960), and this work will be used in § 4 to indicate how the disturbance will spread.

We note finally that, although the estimate of λ contains serious uncertainties, the conclusions do not depend strongly on it. Because loss of a particle's momentum occurs in small bits, as a result of a large number of encounters of rather long range, a somewhat larger λ would mean, not the breakdown of behaviour as a continuous fluid, but simply a larger effective viscosity, which as we shall see does not substantially alter the conclusions.

4. Wave pattern of a moving body

We saw in § 2 that bodies may be expected to excite longitudinal waves along magnetic lines of force. The waves are propagated, according to continuous-fluid theory, as sound waves, and confined within magnetic tubes of force by small currents, the forces on which in the presence of the magnetic field balance the lateral pressure gradients. From the molecular point of view, the wave is propagated by electrostatic forces between charged particles when they come

* The waves propagating δ are absent because, for the characteristic frequency V/l of a body of velocity V and length l, their wavelength is $a_1 l/V$, which is very large compared with the size l of the source. (In the language of Lighthill 1960, § 7, F is small in the relevant range of wave-number.) The waves propagating ξ are probably absent, notably because the body will not generate vorticity, at least if one can represent it by a distribution of sources in the usual way.

close together, and remains one-dimensional because all the charged particles are spiralling tightly about magnetic lines of force, shifting the axes of their spirals only slightly at any encounter.

The value of the sound speed is uncertain, but we take it to be 1200 m/sec, corresponding to a gas consisting equally of electrons and O^+ ions at a temperature of 1100 °K, and with an effective γ of $\frac{4}{3}$, owing to the ion γ being $\frac{5}{3}$ and the electrons behaving isothermally ($\gamma = 1$) because their r.m.s. velocity is 170 times as great.

Another uncertainty is in the type of pulse sent out by the motion of the body. At ordinary altitudes this is an 'N-wave', the compression sent out by the front of the missile being followed by a rarefaction from where it closes up at the rear. It is likely, however, that out-gassing will produce a wake with above-ambient density—which, indeed, might 'balloon out' to form a disturbance much fatter than the body itself—and therefore a pure compression pulse can be regarded as probable.

In this case, the author (Lighthill 1956, § 10.2) has shown that the progress of the pulse depends on a so-called Reynolds number R, defined as the integral over the length of the pulse of a certain velocity characteristic of the magnitude of the disturbance, divided by a diffusivity somewhat greater than the kinematic viscosity ν. To a sufficient approximation we may take R as equal to

$$\frac{(\text{missile velocity}) \times (\text{missile length})}{(\text{sound velocity}) \times (\text{mean free path})}. \tag{6}$$

It is of order 1 in the present application, which signifies roughly equal importance for waveform-steepening and waveform-diffusing effects; for the lower missile velocities R is less than 1, so that diffusion is more important. As time goes on, the Reynolds number of a one-dimensionally propagated pulse remains unchanged, the length increasing asymptotically like $t^{\frac{1}{2}}$ while the amplitude decreases like $t^{-\frac{1}{2}}$. The shape is asymptotically triangular (thin shock wave followed by a linear density decrease) for $R \gg 1$, Gaussian for $R \ll 1$ and intermediate between these two for R of order 1.

These considerations enable us to sketch the wave pattern of bodies moving at velocities 1·5 and 7·5 km/sec in figure 2. The magnetic lines of force are taken to be at 30° to the flight path, and the speed of propagation of waves along them is 0·8 and 0·16 times the missile speed in the two cases. The pulses carry substantial local increases in electron density, which could cause reflexion of a substantial band of radio waves—with frequencies between the critical frequency within the pulse and that in the undisturbed F_2 layer.

The one-dimensional character of the propagation is an essential feature of the above discussion. With cylindrical or spherical pulses an N-wave necessarily forms, the Reynolds number progressively decreases, and after it has fallen below 1 the whole wave decays exponentially.

The present theory yields a rather substantial area of increased electron density viewed head-on as in figure 2, that is to say, viewed at right angles both to the flight path and the magnetic lines of force. However, the thickness of this area (at right angles to the paper in figure 2) is rather small—comparable with the body dimensions, except in so far as these may be effectively increased by ballooning of the wake—owing to the strong restrictions on movement of electrons and ions perpendicular to magnetic lines of force, so that possible echoing area in other directions is accordingly reduced.

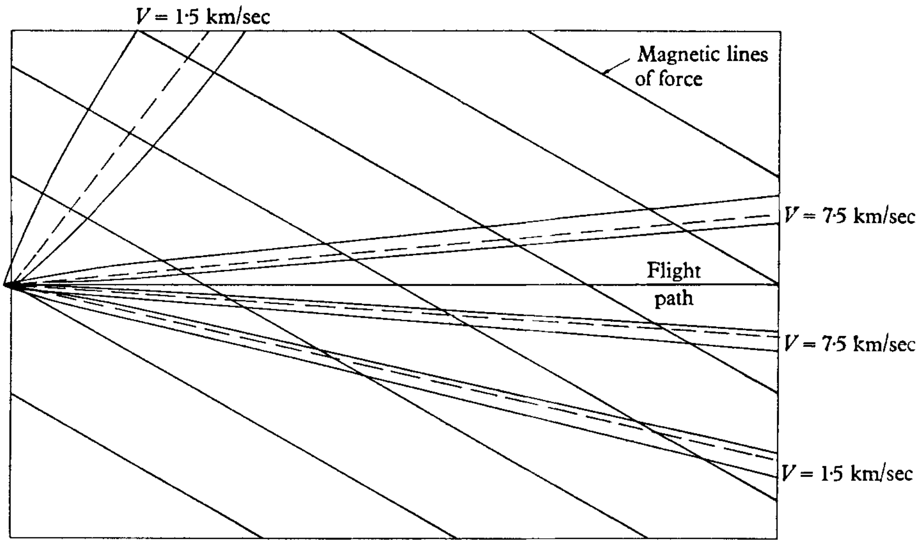

FIGURE 2. Possible compression waves due to a body moving at velocity V equal to (i) 1·5 km/sec, (ii) 7·5 km/sec. The broken lines show where the disturbance would reach to if it were propagated along the magnetic lines of force without any spreading at a velocity of 1·2 km/sec, and the plain curves show how it might spread due to waveform steepening and diffusion.

The above general considerations on the nature of the disturbances that may arise are put forward in the hope of stimulating further discussions and observations that may lead to clearer understanding of the phenomena.

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